

Chapter 1

Spatial beam structures

1.1 Axial load and uniform bending

It is preliminarily noted that the elementary extensional-flexural solution is exact with respect to the Theory of Elasticity if the following conditions hold:

- beam constant section;
- beam rectilinear axis;
- absence of locally applied loads;
- absence of shear resultants¹ (i.e. constant bending moments);
- principal material directions of orthotropy are uniform along the section, and one of them is aligned with the beam axis;
- the ν_{31} and the ν_{32} Poisson’s ratios² are constant along the section, where 3 means the principal direction of orthotropy aligned with the axis. Please note that $E_i\nu_{ji} = E_j\nu_{ij}$, and hence $\nu_{ji} \neq \nu_{ij}$ for a generally orthotropic material.

Most of the above conditions are in fact violated in many textbook structural calculations, thus suggesting that the elementary beam theory is robust enough to be adapted to practical applications, i.e. limited error is expected if some laxity is used in circumscribing its scope³.

The extensional-flexural solution builds on the basis of the following simplifying assumptions:

- the in-plane⁴ stress components $\sigma_x, \sigma_y, \tau_{xy}$ are null;
- the out-of-plane shear stresses τ_{yz}, τ_{zx} are also null;

¹A locally pure shear solution may be in fact superposed; such solution may however not be available for a general cross section.

²We recall that ν_{ij} is the Poisson’s ratio that corresponds to a contraction in direction j , being a unitary extension applied in direction i in a manner that the elastic body is subject to a uniaxial stress state.

³Measures for both the error and the violation have to be supplied first in order to quantify the approximation.

⁴Both the *in-plane* and the *out-of-plane* expressions for the characterization of the stress/strain components refer to the cross sectional plane.

- the axial elongation ϵ_z linearly varies along the cross section, namely

$$\epsilon_z = a + bx + cy \quad (1.1)$$

or, equivalently⁵, each cross section is assumed to remain planar in the deformed configuration.

The three general constants a , b and c possess a physical meaning; in particular a represents the axial elongation $\bar{\epsilon}$ as measured at the centroid⁶, c represents the $1/\rho_x$ curvature⁷ whereas b represent the $1/\rho_y$ curvature, apart from its sign.

Figure 1.1 (c) justifies the equality relation $c = 1/\rho_x$; the beam axial fibers with a Δz initial length are elongated by the curvature up to a $\Delta\theta(\rho_x + y)$ deformed length, where $\Delta\theta\rho_x$ equates Δz based on the length of the unextended fibre at the centroid. By evaluating the axial strain value for a general fiber, it follows that $\epsilon_z = 1/\rho_x y$.

In addition, Figure 1.1 (c) relates the $1/\rho_x$ curvature to the displacement component in the local y direction, namely v , and to the section rotation angle with respect to the local x axis, namely θ , thus obtaining

$$\frac{d\theta}{dz} = \frac{1}{\rho_x}, \quad \theta = -\frac{dv}{dz}, \quad \frac{d^2v}{dz^2} = -\frac{1}{\rho_x} \quad (1.2)$$

Following analogous considerations, see 1.1 (e), we may similarly obtain

$$\frac{d\phi}{dz} = \frac{1}{\rho_y}, \quad \phi = +\frac{du}{dz}, \quad \frac{d^2u}{dz^2} = +\frac{1}{\rho_y} \quad (1.3)$$

where ϕ is the cross section rotation about the local y axis, and u is the x displacement component.

According to the assumptions in the preamble, a uniaxial stress state is assumed, where the only nonzero σ_z stress component may be determined as

$$\sigma_z = E_z \epsilon_z = E_z \left(\bar{\epsilon} - \frac{1}{\rho_y} x + \frac{1}{\rho_x} y \right) \quad (1.4)$$

⁵The axial, out-of-plane displacement $\Delta w = \int_{\Delta l} \epsilon_z dz = \Delta l (a + bx + cy)$ accumulated between two contiguous cross sections with an Δl initial distance, is consistent with that of a relative rigid body motion.

⁶or, equivalently, the average elongation along the section, in an integral sense.

⁷namely the inverse of the beam curvature radii as observed with a line of sight aligned with the x axis. Curvature is assumed positive if the associated θ section rotation grows with increasing z , i.e. $d\theta/dz > 0$.

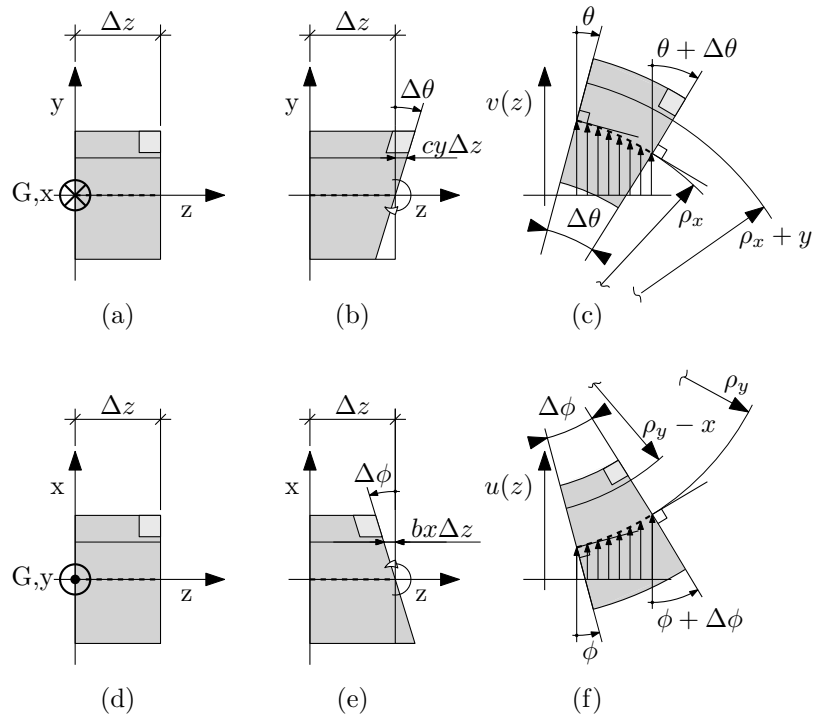


Figure 1.1: A differential fibre elongation proportional to the y coordinate induces a curvature $1/\rho_x$ on the normal plane with respect to the x axis. A differential fibre contraction proportional to the x coordinate induces a curvature $1/\rho_y$ on the normal plane with respect to the y axis. The didascalical trapezoidal deformation modes (b) and (e) clearly associate the differential elongation/contraction with the positive relative end rotation; they are however affected by a spurious shear deformation as evidenced by the skewed corner.

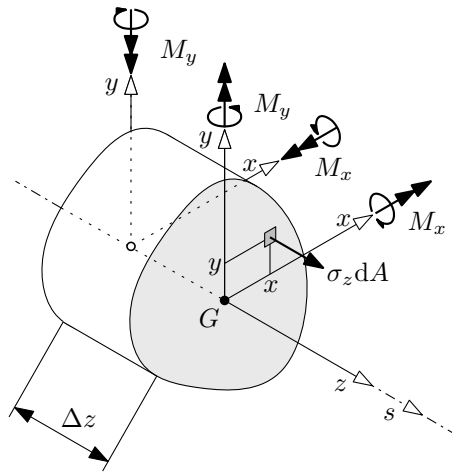


Figure 1.2: Positive x and y bending moment components adopt the same direction of the associated local axes at the beam segment end showing an outward-oriented arclength coordinate axis; at beam segment ends characterized by an inward-oriented local z axis, the same positive bending moment components are locally counter-oriented to the respective axes.

Stress resultants may easily be evaluated based on Fig. 1.1 as

$$N = \iint_{\mathcal{A}} E_z \epsilon_z dA = \overline{EA} \bar{\epsilon} \quad (1.5)$$

$$\mathcal{M}_x = \iint_{\mathcal{A}} E_z \epsilon_z y dA = \overline{EJ}_{xx} \frac{1}{\rho_x} - \overline{EJ}_{xy} \frac{1}{\rho_y} \quad (1.6)$$

$$\mathcal{M}_y = - \iint_{\mathcal{A}} E_z \epsilon_z x dA = -\overline{EJ}_{xy} \frac{1}{\rho_x} + \overline{EJ}_{yy} \frac{1}{\rho_y} \quad (1.7)$$

where the combined material/cross-section stiffness moduli

$$\overline{EA} = \iint_{\mathcal{A}} E_z(x, y) dA \quad (1.8)$$

$$\overline{EJ}_{xx} = \iint_{\mathcal{A}} E_z(x, y) yy dA \quad (1.9)$$

$$\overline{EJ}_{xy} = \iint_{\mathcal{A}} E_z(x, y) yx dA \quad (1.10)$$

$$\overline{EJ}_{yy} = \iint_{\mathcal{A}} E_z(x, y) xx dA \quad (1.11)$$

may also be rationalized as the cross section area and moment of inertia, respectively, multiplied by a suitably averaged Young modulus, evaluated in the axial direction.

Those moduli simplify to their usual $E_z A, E_z J_{**}$ analogues, where the influence of the material and of the geometry are separated if the former is homogeneous along the beam cross section.

From Eqn. 1.5 we obtain

$$\bar{\epsilon} = \frac{N}{EA}. \quad (1.12)$$

By concurrently solving Eqns. 1.6 and 1.7 with respect to the $1/\rho_x$ and $1/\rho_y$ curvatures, we obtain

$$\frac{1}{\rho_x} = \frac{\mathcal{M}_x \overline{EJ}_{yy} + \mathcal{M}_y \overline{EJ}_{xy}}{\overline{EJ}_{xx} \overline{EJ}_{yy} - \overline{EJ}_{xy}^2} \quad (1.13)$$

$$\frac{1}{\rho_y} = \frac{\mathcal{M}_x \overline{EJ}_{xy} + \mathcal{M}_y \overline{EJ}_{xx}}{\overline{EJ}_{xx} \overline{EJ}_{yy} - \overline{EJ}_{xy}^2} \quad (1.14)$$

Axial strain and stress components may then be obtained for any cross section point by substituting the above calculated generalized strain components $\bar{\epsilon}, 1/\rho_x$ and $1/\rho_y$ holding for the extensional-flexural beam into Eqn. 1.4, thus obtaining

$$\sigma_z = E_z \epsilon_z \quad (1.15)$$

$$= \alpha \mathcal{M}_x + \beta \mathcal{M}_y + \gamma N \quad (1.16)$$

where

$$\alpha(x, y, E_z, \overline{EJ}_{**}) = E_z(x, y) \frac{-\overline{EJ}_{xy}x + \overline{EJ}_{yy}y}{\overline{EJ}_{xx}\overline{EJ}_{yy} - \overline{EJ}_{xy}^2} \quad (1.17)$$

$$\beta(x, y, E_z, \overline{EJ}_{**}) = E_z(x, y) \frac{-\overline{EJ}_{xx}x + \overline{EJ}_{xy}y}{\overline{EJ}_{xx}\overline{EJ}_{yy} - \overline{EJ}_{xy}^2} \quad (1.18)$$

$$\gamma(x, y, E_z, \overline{EA}) = E_z(x, y) \frac{1}{\overline{EA}}. \quad (1.19)$$

The peak axial strain is obtained at points farther from neutral axis of the stretched section; such neutral axis may be graphically defined as follows:

- nonzero $\bar{\epsilon}$ case: the neutral axis intersect the local axes at the $(x, y) \equiv (0, -\bar{\epsilon}\rho_x)$ and $(\bar{\epsilon}\rho_y, 0)$ intercepts. A divergent intercept with respect to one axis denotes parallelism;
- zero $\bar{\epsilon}$ case: the neutral axis is centroidal and parametrically defined by the $\lambda(1/\rho_x, 1/\rho_y)$ points, with arbitrary λ .

In both cases, the direction that is normal to the neutral axis is parametrically defined as $\lambda(-1/\rho_y, +1/\rho_x)$. Elongation increases with growing λ . The cross section projection on such a line defines a segment whose ends are extremal with respect to the axial strain.

Equivalently, we may parametrically define the neutral axis as

$$(x, y) \equiv \left(\underbrace{\frac{\bar{\epsilon}\rho_x^2\rho_y}{\rho_x^2 + \rho_y^2}}_{x_N} \pm r \frac{\rho_y}{\sqrt{\rho_x^2 + \rho_y^2}}, -\underbrace{\frac{\bar{\epsilon}\rho_x\rho_y^2}{\rho_x^2 + \rho_y^2}}_{y_N} \pm r \frac{\rho_x}{\sqrt{\rho_x^2 + \rho_y^2}} \right)$$

where (x_N, y_N) the nearest pass-through point with respect to the G centroid, and the r parameter spans the axis as the distance from such a point⁸.

⁸XXX

1.2 Shear stresses due to the St. Venant torsion

The classical solution for the rectilinear beam subject to uniform torsion predicts a displacement field that is composed by the superposition of a) a rigid, in-plane⁹ cross section rotation about the shear centre, named twist, of uniform axial rate, and b) an out-of-plane *warping* displacement that is uniform in the axial direction, whereas it varies within the section; such warping displacement is zero in the case of axisymmetric sections only (e.g. hollow or solid circular cross sections).

In-plane stress component σ_x , σ_y , τ_{xy} are assumed zero, along with the normal stress σ_z . The motion is internally restricted only due to the nonzero out-of-plane shear stresses τ_{yz} and τ_{zx} , that develop as an elastic reaction to the associated strain components.

A more in-depth treatise of the topic involves the solution of an plane, inhomogeneous Laplace differential equation with essential conditions imposed at the cross section boundary, which is beyond the scope of the present contribution.

However, in the case of open- and closed- section, thin walled beams, simplified solution are available based on the assumptions that a) the out-of-plane shear stresses are locally aligned to the wall midsurface - i.e. $\tau_{zr} = 0$ leaving τ_{zs} as the only nonzero stress component¹⁰, and b) the residual τ_{zs} shear component is either constant by moving through the wall thickness (closed section case), or it linearly varies with the through-thickness coordinate r .

1.2.1 Solid section beam

TODO.

1.2.2 Closed section, thin walled beam

TODO.

⁹the rotation vector is actually normal to the cross sectional plane; the *in-plane* motion characterization refers to the associated displacement field.

¹⁰Here, the notation introduced in paragraph XXX for the thin walled section is employed.

1.2.3 Open section, thin walled beam

For thin enough open sections of uniform and isotropic material we have

$$K_T \approx \frac{1}{3} \int_0^l t^3(s) ds \quad (1.20)$$

If the thin-walled cross section may be described as a sequence of constant thickness wall segments, the simplified formula

$$K_T \approx \frac{1}{3} \sum_i l_i t_i^3 \quad (1.21)$$

is obtained where t_i and l_i are respectively the length and the thickness of each segment.

1.3 Stresses due to the shear cross section resultants

In the presence of nonzero shear resultants, the bending moment exhibits a linear variation with the axial coordinate z in a straight beam. Based on the beam segment equilibrium we have

$$S_y = \frac{d\mathcal{M}_x}{dz}, \quad S_x = -\frac{d\mathcal{M}_y}{dz}, \quad (1.22)$$

as rationalized in Fig. XXX (a), with $z \rightarrow 0$ and differentiable $\mathcal{M}_x, \mathcal{M}_y$ with respect to z .

The linear variation of the bending moment in z induces an equally linear variation of the pointwise axial stress due to bending. In particular, the differentiation with respect to z of Eqn.1.16 returns

$$\frac{d\sigma_z}{dz} = \alpha(x, y, E_z, \overline{EJ}_{**}) S_y - \beta(x, y, E_z, \overline{EJ}_{**}) S_x \quad (1.23)$$

since its α, β, γ factors are constant with respect to the bending moments¹¹; the bending moment derivatives are here expressed as a function of the shear actions, as in Eqns. 1.22.

¹¹The pointwise axial stress is a linear function of the two bending moments

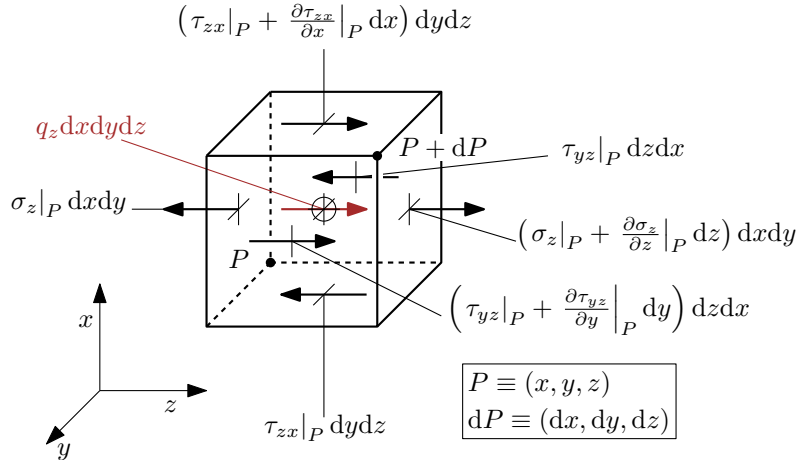


Figure 1.3: Equilibrium conditions with respect to the axial z translation for the infinitesimal volume extracted from the beam. In the case under scrutiny, the distributed volume action q_z is null.

Figure 1.3 rationalizes the axial equilibrium for an elementary volume of material; we have

$$\frac{d\tau_{zx}}{dx} + \frac{d\tau_{yz}}{dy} + \frac{d\sigma_z}{dz} + q_z = 0 \quad (1.24)$$

where, for the specific case, the distributed volumetric load q_z is zero. It clearly derives from such relation that the shear stresses τ_{zx}, τ_{yz} that were null within the uniform bending framework are non-uniform along the section, and hence nonzero, in the presence of shear resultants.

A treatise on the pointwise solution of a) Eqn. 1.24, once coupled with b) the compatibility conditions and with c) the the material elastic response is beyond the scope of the present contribution.

1.3.1 The Jourawsky approach and its extension for a general section

The aforementioned axial equilibrium condition, whose solution is difficult for the infinitesimal volume, may be more conveniently dealt with if a finite portion of the beam segment is taken into account, as in Figure 1.3.1.

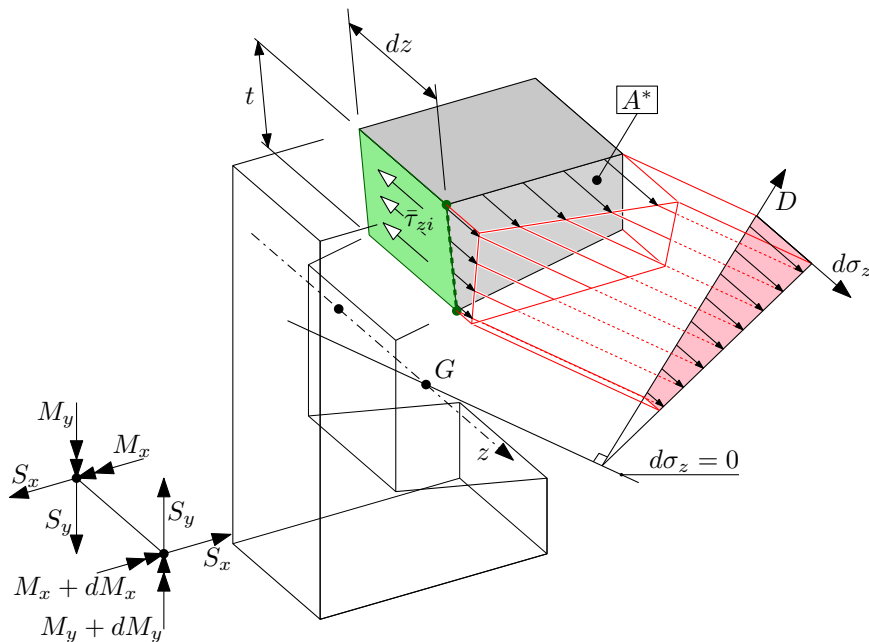


Figure 1.4: Equilibrium conditions for the isolated beam segment portion. It is noted that the null σ_z variation locus, $d\sigma_z = 0$, does not coincide with the bending neutral axis in general. Also, the depicted linear variation of $d\sigma_z$ with the D distance from such null $d\sigma_z$ locus does not hold in the case of non-uniform E_z modulus.

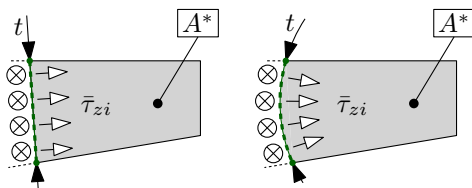


Figure 1.5: XXX.

The beam cross section is partitioned based on a (possibly curve, see Fig. 1.3.1) line that isolates an area portion A^* for further scrutiny. A beam segment is considered whose axial extent is dz , and a portion of such beam segment may be isolated accordingly; an axial equilibrium equation may then be stated for the isolated portion as follows

$$\bar{\tau}_{zi}t = \int_{A^*} \frac{d\sigma_z}{dz} dA, \quad (1.25)$$

where

$$\bar{\tau}_{zi} = \frac{1}{t} \int_t \tau_{zi} dr \quad (1.26)$$

is the average shear stress acting in the z direction along the cutting surface; i is the (locally normal) inward direction with respect to such a surface. Due to the reciprocal nature of the shear stresses, the same $\bar{\tau}_{zi}$ shear stress acts along the cross sectional plane, and locally at the cutting curve itself. Those shear actions are assumed positive if inward directed with respect to A^* .

The $\bar{\tau}_{zi}t$ is named *shear flow*, and may be evaluated along a generic cutting curve.

It is noted that according to Eqn. 1.25 no information is provided with regard to a) the τ_{zt} shear stress that acts parallel to the cutting curve, nor b) the pointwise variation of τ_{zi} with respect of its average value $\bar{\tau}_{zi}$. Those quantities may just be neglected without resorting to more cumbersome calculation frameworks; an informed choice for the cutting curve is thus critical for a reliable application of the method, being optimal the adoption of the (normally unknown) iso-shear stress lines of the exact solution.

In the simplified case of a) uniform material and b) local x, y axes that are principal axes of inertia (i.e. $J_{xy} = 0$), the usual formula is obtained

$$\bar{\tau}_{zi}t = \int_{A^*} \left(\frac{yS_y}{J_{xx}} + \frac{xS_x}{J_{yy}} \right) dA = \frac{\bar{y}^* A^*}{J_{xx}} S_y + \frac{\bar{x}^* A^*}{J_{yy}} S_x, \quad (1.27)$$

where $\bar{y}^* A^*$ and $\bar{x}^* A^*$ are the first order area moments of the A^* section portion with respect to the x and y axes respectively¹².

¹²According to the employed notation, (\bar{x}^*, \bar{y}^*) are the center of gravity coordinates for the A^* area.

1.3.2 Shear stresses in an open section, thin walled beam

TODO.

1.3.3 Shear stresses in an closed section, thin walled beam

TODO.

1.4 Castigliano’s second theorem and its applications

Castigliano’s second theorem may be employed for calculating deflections and rotations, and it states:

If the strain energy of an elastic structure can be expressed as a function of generalised loads Q_i (namely, forces or moments) then the partial derivative of the strain energy with respect to generalised forces supplies the generalised displacement q_i (namely displacements and rotations with respect to which the generalized forces work).

In equation form,

$$q_i = \frac{\partial U}{\partial Q_i}$$

where U is the strain energy.

1.5 Internal energy for the spatial straight beam

The linear density of the elastic potential (alternatively named internal) energy for the spatial rectilinear beam may be derived as a function of its cross section resultants, namely

$$\frac{dU}{dl} = \frac{J_{\eta\eta}M_\xi^2 + J_{\xi\xi}M_\eta^2 + 2J_{\xi\eta}M_\xi M_\eta}{2E(J_{\xi\xi}J_{\eta\eta} - J_{\xi\eta}^2)} + \frac{N^2}{2EA} \quad (1.28)$$

$$+ \frac{\chi_\xi S_\xi^2 + \chi_\eta S_\eta^2 + \chi_{\xi\eta}S_\eta S_\xi}{2GA} + \frac{M_t^2}{2GK_t} \quad (1.29)$$

where

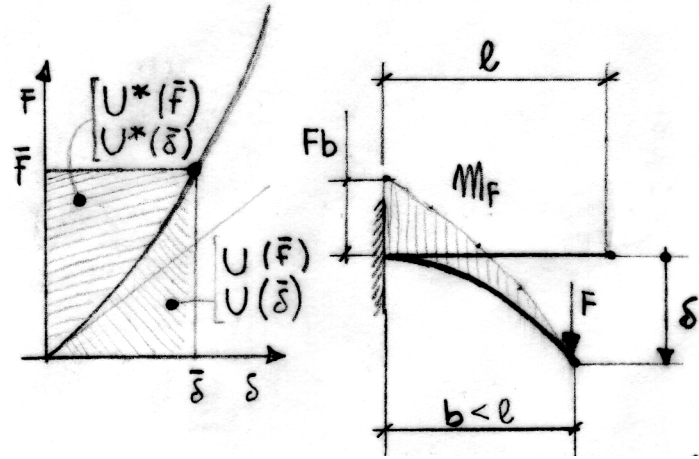


Figure 1.6: A nonlinearly elastic (namely stiffening) structure; the bending moment diagram is evaluated based on the beam portion equilibrium in its *deformed* configuration. The complementary elastic strain energy U^* is plotted for a given applied load \bar{f} or assumed displacement $\bar{\delta}$, alongside the elastic strain energy U .

- $A, J_{\eta\eta}, J_{\xi\xi}$ and $J_{\xi\eta}$ are the section area and moments of inertia, respectively;
- K_t is the section torsional stiffness (**not** generally equivalent to its polar moment of inertia);
- E and G are the material Young Modulus and Shear Modulus, respectively; the material is assumed homogeneous, isotropic and linearly elastic.

The shear energy normalized coefficients $\chi_\eta, \chi_\xi, \chi_{\xi\eta}$ are specific to the cross section geometry, and may be collected from the expression of the actual shear strain energy due to concurrent action of the S_η, S_ξ shear forces.

In cases of elastically nonlinear structures, the second Castigliano theorem may still be employed, provided that the complementary elastic strain energy U^* is employed in place of the strain energy U , see Fig. 1.5. The two energy terms are equal for linearly behaving structures.

