Chapter 1

Spatial beam structures

1.1 Beam axis and cross section definition

A basic necessary condition for identifying a deformable body as a beam – and hence applying the associated framework – is that its centroidal curve be at least loosely recognizable.

Once such centroidal line has been roughly defined, locally perpendicular planes may be derived whose intersection with the body itself defines the local beam cross section.

Then, the G center of gravity position may be computed for each of the local cross sections, leading to a refined, potentially iterative definition for the beam centroidal axis[1].

A local cross-sectional reference system may be defined by aligning the normal z axis with the beam centroidal curve, and by employing, as the first in-section axis, namely x, the projection of a given global $\underline{\mathrm{v}}$ vector, which is assumed not to be parallel to the beam axis.

The second in-section axis y may be then derived, in order to obtain a $Gxyz$ right-handed coordinate system, whose unit vectors are $\hat{\imath}, \hat{\jmath}, \hat{k}$.

If a thin walled profile is considered in place of a solid cross section member – i.e., the section wall midplane is recognizable too (see paragraph XXX), then a curvilinear coordinate $0 \leq s \leq l$ may be defined that spans the in-cross-section wall midplane, along with a local through-wall-thickness coordinate $-t(s)/2 \leq r \leq +t(s)/2$.

Such s, r, in-section coordinates based on the profile wall may be employed in place of their cartesian x, y counterparts, if favourable.

Beam axis may be discontinuous at sudden body geometry changes; a rigid body connection is ideally assumed to restrict the relative motion of the proximal segments. Such rigid joint modeling may be extended to more complex n-way joints; if the joint finite stiffness is to be taken into account, it has to be described through the entries of a rank $6(n-1)$ symmetric square matrix [2].

At joints or beam axis angular points the cylindrical bodies swept by the cross sections do usually overlap, besides they only loosely mimic the actual deformable body geometry; the results obtained through the local application of the elementary beam theory may at most be

[1] Here, centroidal curve, centroidal line, centroidal axis, or simply beam axis are treated as synonyms.

[2] i.e., joint stiffness is unfortunately not a scalar value.

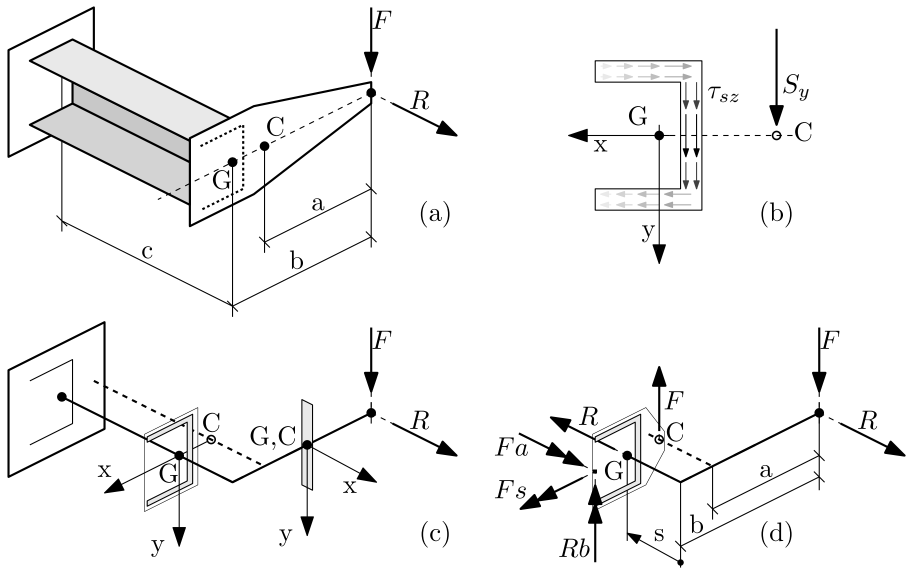

Figure 1.1: A beam structure.

employed to scale the triaxial local stress/strain fields[3], which have to be evaluated resorting to more complex modelings.

1.1.1 A worked example

See Figure 1.1.1. TODO.

1.2 Cross-sectional resultants for the spatial beam

At any point along the axis the beam may be notionally split, thus obtaining two facing cross sections, whose interaction is limited to three components of interfacial stresses, namely the axial normal stress σ_{zz} and the two shear components τ_{yz}, τ_{zx}.

Three force resultant components may be defined by integration along the cross section area, namely the normal force, the y- and the

[3]The peak stress values obtained through the elementary beam theory may be profitably employed as *nominal stresses* within the stress concentration effect framework.

x- oriented shear forces, respectively defined as

$$N = \int_{\mathcal{A}} \sigma_{zz} d\mathcal{A}$$
$$S_y = \int_{\mathcal{A}} \tau_{yz} d\mathcal{A}$$
$$S_x = \int_{\mathcal{A}} \tau_{zx} d\mathcal{A}$$

Three moment resultant components may be similarly defined, namely the x- and y- oriented bending moments, and the torsional moment. However, if the centroid is the preferred fulcrum for evaluating the bending moments, the below discussed C shear center is employed for evaluating the torsional moment. We hence define

$$\mathcal{M}_x = \int_{\mathcal{A}} \sigma_{zz} y d\mathcal{A}$$
$$\mathcal{M}_y = -\int_{\mathcal{A}} \sigma_{zz} x d\mathcal{A}$$
$$\mathcal{M}_t \equiv \mathcal{M}_z = \int_{\mathcal{A}} \left[\tau_{yz}(x - x_C) - \tau_{zx}(y - y_C)\right] d\mathcal{A}$$

The applied vector associated to the normal force component $(G, N\hat{k})$ is located at the section center of gravity , whereas the shear force $(C, S_x\hat{\imath} + S_y\hat{\jmath})$ is supposed to act at the shear center; such convention decouples the energy contribution of force and moment components for the straight beam.

Cross section resultants may be obtained, based on equilibrium for a statically determinate structure. The ordinary procedure consists in

- notionally splitting the structure at the cross section whose resultants are under scrutiny;
- isolating a portion of the structure that ends at the cut, whose locally applied loads are all known; the structure has to be preliminarily solved for the all the constraint reactions that act on the isolated portion;
- setting the equilibrium equations for the isolated substructure, according to which the cross-sectional resultants are in equilibrium with whole loading.

1.3 Axial load and uniform bending

It is preliminarily noted that the elementary extensional-flexural solution is exact with respect to the Theory of Elasticity if the following conditions hold:

- beam constant section;
- beam rectilinear axis;
- absence of locally applied loads;
- absence of shear resultants[4] (i.e. constant bending moments);
- principal material directions of orthotropy are uniform along the section, and one of them is aligned with the beam axis;
- the ν_{31} and the ν_{32} Poisson's ratios[5] are constant along the section, where 3 means the principal direction of orthotropy aligned with the axis. Please note that $E_i \nu_{ji} = E_j \nu_{ij}$, and hence $\nu_{ji} \neq \nu_{ij}$ for a generally orthotropic material.

Most of the above conditions are in fact violated in many textbook structural calculations, thus suggesting that the elementary beam theory is robust enough to be adapted to practical applications, i.e. limited error is expected if some laxity is used in circumscribing its scope[6].

The extensional-flexural solution is build on the basis of the following simplifying assumptions:

- the in-plane[7] stress components $\sigma_x, \sigma_y, \tau_{xy}$ are null;
- the out-of-plane shear stresses τ_{yz}, τ_{zx} are also null;

[4]A locally pure shear solution may be in fact superposed; such solution may however not be available for a general cross section.

[5]We recall that ν_{ij} is the Poisson's ratio that corresponds to a contraction in direction j, being a unitary extension applied in direction i in a manner that the elastic body is subject to a uniaxial stress state.

[6]Measures for both the error and the violation have to be supplied first in order to quantify the approximation.

[7]The *in-plane* and the *out-of-plane expression refer to the cross sectional plane.* stress/strain component characterization both refer to the cross sectional plane, if not otherwise specified

- the axial elongation ϵ_z linearly varies along the cross section, namely
$$\epsilon_z = a + bx + cy \tag{1.1}$$
or, equivalently[8], each cross section is assumed to remain planar in the deformed configuration.

The three general constants a, b and c have a physical counterpart; in particular a represents the axial elongation $\bar{\epsilon}$ as measured at the centroid[9], c equates the $1/\rho_x$ curvature[10] whereas b equates $-1/\rho_y$.

Figure 1.3 (c) justifies the equality relation $c = 1/\rho_x$; beam axial fibers with a Δz initial length are elongated by the curvature up to a $\Delta\theta\,(\rho_x + y)$ deformed length, where $\Delta\theta\rho_x$ equates Δz based on the length of the unextended fibre at the centroid. By evaluating the axial strain value for such general fiber, it results $\epsilon_z = 1/\rho_x\, y$.

In addition, Figure 1.3 (c) relates the $1/\rho_x$ curvature with the displacement component in the local y direction, namely v, and with the section rotation angle with respect to the local x axis, namely θ, thus obtaining
$$\frac{d\theta}{dz} = \frac{1}{\rho_x}, \quad \theta = -\frac{dv}{dz}, \quad \frac{d^2v}{dz^2} = -\frac{1}{\rho_x} \tag{1.2}$$
With analogous considerations, see 1.3 (e), we may also obtain
$$\frac{d\phi}{dz} = \frac{1}{\rho_y}, \quad \phi = +\frac{du}{dz}, \quad \frac{d^2u}{dz^2} = +\frac{1}{\rho_y} \tag{1.3}$$
where ϕ is the cross section rotation around the local y axis, and u is the x displacement component.

A uniaxial stress state is hence assumed, where the only nonzero stress component may be determined as
$$\sigma_z = E_z\epsilon_z = E_z\left(\bar{\epsilon} - \frac{1}{\rho_y}x + \frac{1}{\rho_x}y\right) \tag{1.4}$$

[8]The axial, out-of-plane displacement $\Delta w = \int_{\Delta l} \epsilon_z dz = \Delta_l\,(a + bx + cy)$ accumulated in the between two cross sections with a Δl initial distance, is consistent with that of a relative rigid body motion.

[9]or, equivalently, the integral average of the elongation along the section.

[10]namely the inverse of the beam curvature radii as observed with a line of sight aligned with the x axis. Curvature is assumed positive if the θ section rotation with respect to the x axis grows with increasing z, i.e. $d\theta/dz > 0$.

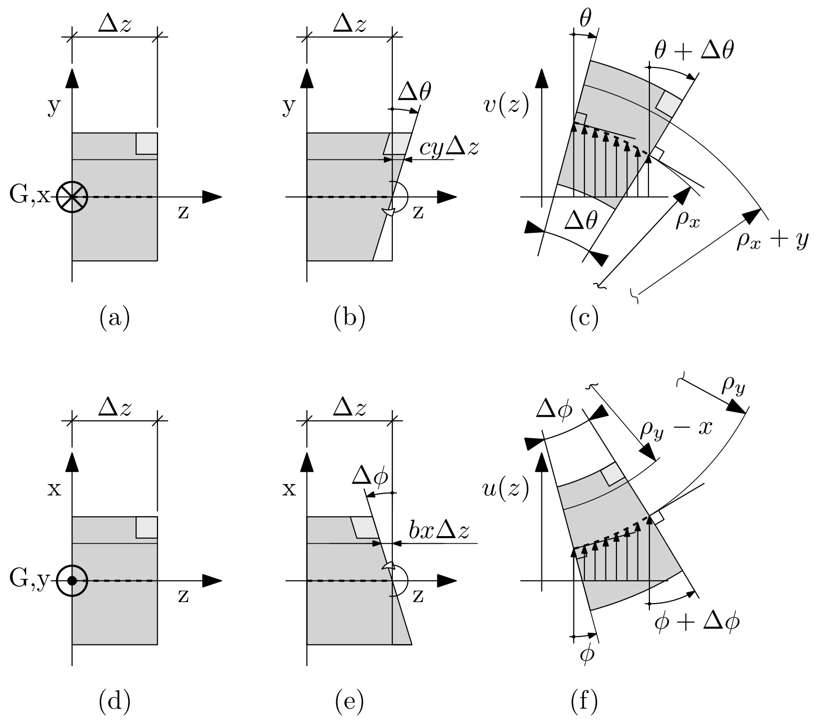

Figure 1.2: A differential fibre elongation proportional to the y coordinate induces a curvature $1/\rho_x$ on the normal plane with respect to the x axis. A differential fibre contraction proportional to the x coordinate induces a curvature $1/\rho_y$ on the normal plane with respect to the y axis. The intermediate trapezoidal deformation modes (b) and (e) clearly relate the differential elongation/contraction and the positive relative end rotation; they are however affected by spurious shear deformation as evidenced by the skewed corner.

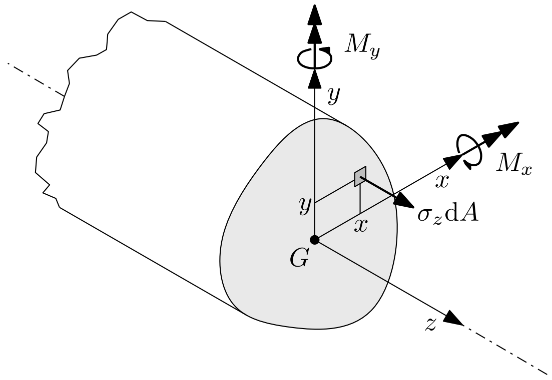

Figure 1.3: Positive x and y bending moment components adopt the same direction of the associated local axes at the beam segment end showing an outward-oriented arclength coordinate axis; at beam segment ends characterized by an inward-oriented local z axis, an opposite sign convention holds for the bending moments.

Stress resultants may be easily evaluated based on Fig. 1.3 as

$$N = \iint_{\mathcal{A}} E_z(x, y)\epsilon_z dA = \overline{EA}\bar{\epsilon} \tag{1.5}$$

$$\mathcal{M}_x = \iint_{\mathcal{A}} E_z(x, y)\epsilon_z y dA = \overline{EJ}_{xx}\frac{1}{\rho_x} - \overline{EJ}_{xy}\frac{1}{\rho_y} \tag{1.6}$$

$$\mathcal{M}_y = -\iint_{\mathcal{A}} E_z(x, y)\epsilon_z x dA = -\overline{EJ}_{xy}\frac{1}{\rho_x} + \overline{EJ}_{yy}\frac{1}{\rho_y} \tag{1.7}$$

where the combined material/cross-section stiffness moduli

$$\overline{EA} = \iint_{\mathcal{A}} E_z(x, y)\; dA \tag{1.8}$$

$$\overline{EJ}_{**} = \iint_{\mathcal{A}} E_z(x, y) * *\, dA \tag{1.9}$$

may also be rationalized as the cross section area and moment of inertia respectively, multiplied by a suitably weighted average axial Young modulus. Those moduli reduce to their usual $E_z A, E_z J_{**}$ counterparts if the material is homogeneous along the cross section.

From Eqn. 1.5 we obtain

$$\bar{\epsilon} = \frac{N}{\overline{EA}}, \tag{1.10}$$

and by concurrently solving Eqns. 1.6 and 1.7 with respect to the $1/\rho_x$ and $1/\rho_y$ curvatures, we obtain

$$\frac{1}{\rho_x} = \frac{\mathcal{M}_x \overline{EJ}_{yy} + \mathcal{M}_y \overline{EJ}_{xy}}{\overline{EJ}_{xx}\overline{EJ}_{yy} - \overline{EJ}_{xy}^2} \tag{1.11}$$

$$\frac{1}{\rho_y} = \frac{\mathcal{M}_x \overline{EJ}_{xy} + \mathcal{M}_y \overline{EJ}_{xx}}{\overline{EJ}_{xx}\overline{EJ}_{yy} - \overline{EJ}_{xy}^2} \tag{1.12}$$

Axial strain and stress components may then be obtained for points along the section once recalled Eqn. 1.4.

Peak axial strain is obtained at points whose distance is extremal with respect to the stretched section neutral axis; such neutral axis may be graphically defined as follows:

- nonzero $\bar{\epsilon}$ case: the neutral axis intersect the local axes $(0, -\bar{\epsilon}\rho_x)$ and $(\bar{\epsilon}\rho_y, 0)$ intercepts. A divergent intercept with respect to one axis denotes parallelism;

- zero $\bar{\epsilon}$ case: the neutral axis is centroidal and parametrically defined by the $\lambda(\rho_y, \rho_x)$ points, with arbitrary λ

In both cases, the direction that is normal to the neutral axis is parametrically defined as $\lambda(-\rho_x, \rho_y)$. Elongation increases with growing λ. The cross section projection on such a line defines a segment whose ends are extremal with respect to the axial strain.

1.4 Shear stresses due to the St. Venant torsion

1.4.1 Closed section, thin walled beam

TODO.

1.4.2 Open section, thin walled beam

TODO.

1.5 Stresses due to the shear cross section resultants

1.5.1 A generalized application of the Jourawsky formula

In the presence of nonzero shear resultants, the bending moment exhibits linear variation with the axial coordinate z in a straight beam. Based on the beam segment equilibrium we have

$$S_y = \frac{d\mathcal{M}_x}{dz}, \quad S_x = -\frac{d\mathcal{M}_x}{dz} \tag{1.13}$$

In the case of constant section, Eqns. 1.11 and 1.12, XXX

1.5.2 Shear stresses in an open section, thin walled beam

TODO.

1.5.3 Shear stresses in an closed section, thin walled beam

TODO.

1.6 Symmetry and skew-symmetry conditions

Symmetric and skew-symmetric loading conditions are mostly relevant for linearly-behaving systems; a nonlinear system may develop an asymmetric response to symmetric loading (e.g. column buckling).

Figure 1.6 collects symmetrical and skew-symmetrical pairs of vectors and moment vectors (moments); those (generalized) vectors are applied at symmetric points in space with respect to the reference plane. Normal and parallel to the plane vectors are considered, that may embody the same named components of a general vector.

The pair members may be moved towards the reference plane up to a vanishing distance ϵ; a point on the reference plane coincides with its image. In the case different (in particular, opposite and nonzero) field vectors are associated to the two coincident pair members, single valuedness does not hold at the reference plane; such condition deserves

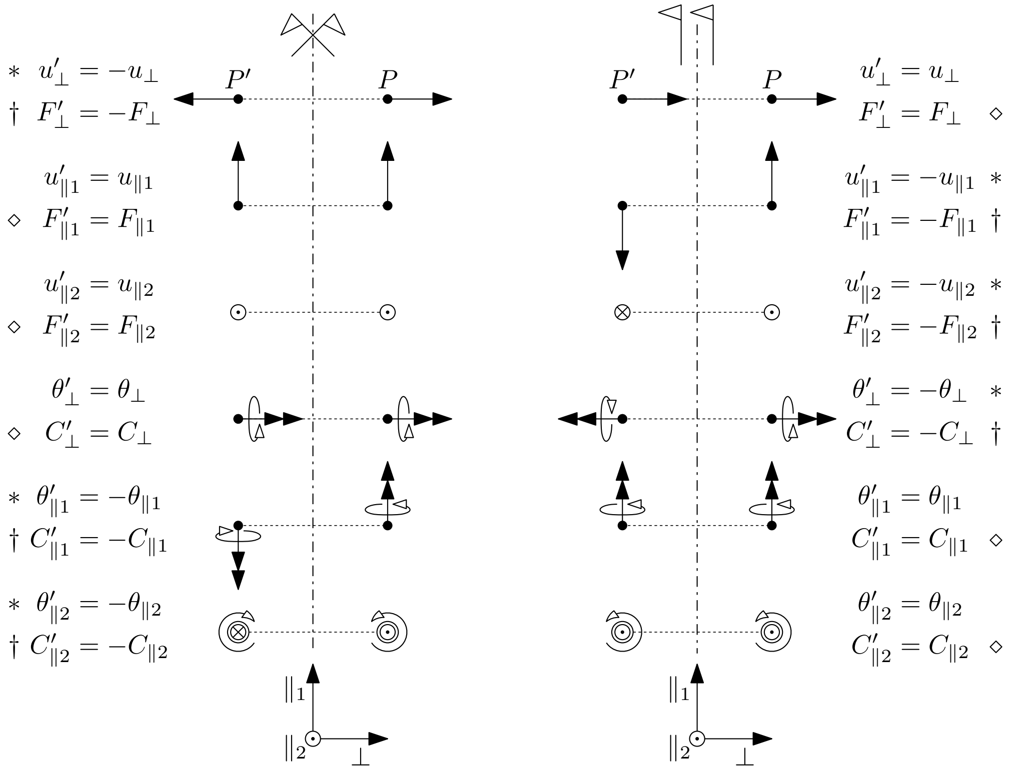

Figure 1.4: An overview of symmetrical and skew-symmetrical (generalized) loading and displacements.

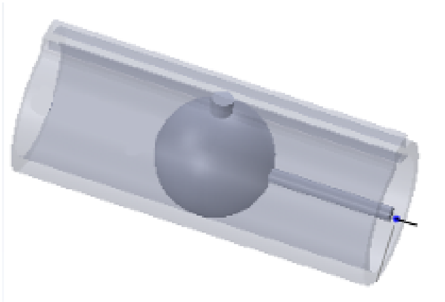

Figure 1.5: The doweled sphere - slotted cylinder joint, which is associated to the skew-symmetry constraint. In this particular application, the cylindrical guide may be considered as grounded.

an attentive rationalization whenever a physical field (displacement field, applied force field, etc.) is to be represented.

Those vector and moment pairs may represent generalized forces (both internal and external) and displacements.

The $*$ (generalized) displacement components may induce material discontinuity at points laying on the [skew-]symmetry plane, if nonzero. They have to be constrained to zero value at those points, thus introducing [skew-]symmetry constraints.

These constraints act in place of the portion of the structure that is omitted from our model, since the results for the whole structure may be derived from the modeled portion alone, due to [skew-]symmetry.

In case of symmetry, a constraint equivalent to a planar joint is to be applied at points laying on the symmetry plane for ensuring displacement/rotation continuity between the modeled portion of the structure, and its image. In case of skew-symmetry, a constraint equivalent to a *doweled sphere - slotted cylinder* joint (see Figure **??**), where the guide axis is orthogonal to the skew-symmetry plane, is applied at the points belonging to the intersection between the deformable body and the plane.

The $\diamond$ internal action components are null at points pertaining to the [skew-]symmetry plane, since they would otherwise violate the action-reaction law. The complementary † internal action components are generally nonzero at the [skew-]symmetry plate.

The † external action components are not allowed at points along the [skew-]symmetry plane; instead, the complementary ⋄ generalized force components are allowed, if they are due to external actions.

In the case of a symmetric structure, generally asymmetric applied loads may be decomposed in a symmetric part and in a skew-symmetric part; the problem may be solved by employing a half structure model for both the loadcases; the results may finally be superposed since the system is assumed linear.

1.7 Periodicity conditions

TODO, if needed.

1.8 Castigliano's second theorem and its applications

Castigliano's second theorem may be employed for calculating deflections and rotations, and it states:

> If the strain energy of an elastic structure can be expressed as a function of generalised loads Q_i (namely, forces or moments) then the partial derivative of the strain energy with respect to generalised forces supplies the generalised displacement q_i (namely displacements and rotations with respect to which the generalized forces work).

In equation form,

$$q_i = \frac{\partial U}{\partial Q_i}$$

where U is the strain energy.

1.9 Internal energy for the spatial straight beam

The linear density of the elastic potential (alternatively named internal) energy for the spatial rectilinear beam may be derived as a function of its cross section resultants, namely

$$\frac{dU}{dl} = \frac{J_{\eta\eta}M_\xi^2 + J_{\xi\xi}M_\eta^2 + 2J_{\xi\eta}M_\xi M_\eta}{2E\left(J_{\xi\xi}J_{\eta\eta} - J_{\xi\eta}^2\right)} + \frac{N^2}{2EA} \tag{1.14}$$

$$+ \frac{\chi_\xi S_\xi^2 + \chi_\eta S_\eta^2 + \chi_{\xi\eta} S_\eta S_\xi}{2GA} + \frac{M_t^2}{2GK_t} \tag{1.15}$$

where

- A, $J_{\eta\eta}$, $J_{\xi\xi}$ and $J_{\xi\eta}$ are the section area and moments of inertia, respectively;
- K_t is the section torsional stiffness (**not** generally equivalent to its polar moment of inertia);
- E and G are the material Young Modulus and Shear Modulus, respectively; the material is assumed homogeneous, isotropic and linearly elastic.

The shear energy normalized coefficients χ_η,χ_ξ,$\chi_{\xi\eta}$ are specific to the cross section geometry, and may be collected from the expression of the actual shear strain energy due to concurrent action of the S_η, S_ξ shear forces.

In cases of elastically nonlinear structures, the second Castigliano theorem may still be employed provided that the complementary elastic strain energy U^* is employed in place of its classical counterpart, see Fig. 1.9. The two energy terms are equal for linearly behaving structures.

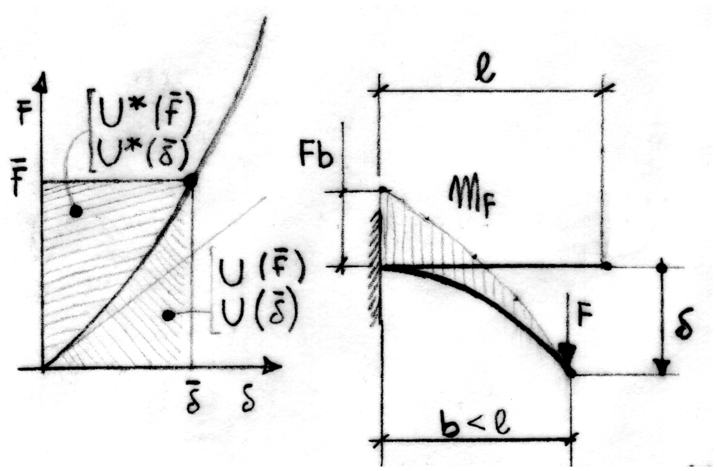

Figure 1.6: A nonlinearly elastic (namely stiffening) structure; the bending moment diagram is evaluated based on the beam portion equilibrium in its *deformed* configuration. The complementary elastic strain energy U^* is plotted for a given applied load $\bar{f}$ or assumed displacement $\bar{\delta}$, alongside the elastic strain energy U.