Chapter 1

Fundamentals of Finite Element Method for structural applications

1.1 Introduction

Briefly. Missing.

1.2 Preparatory formulations

1.2.1 Interpolation functions for some elementary domains

Quadrilateral domain. A quadrilateral domain is considered whose vertices are conventionally located at the $[-1,-1]$, $[1,-1]$, $[1,1]$ and $[-1,1]$ points of an adimensional $[\xi,\eta]$ plane coordinate system, see Figure. Scalar values f_i are associated to a set of *nodal* points $\mathrm{P}_i \equiv [\xi_i,\eta_i]$, which for the present case coincide with the quadrangle vertices, numbered as in figure.

A $f(\xi,\eta)$ interpolation function may be devised by defining a set of nodal influence functions $N_i(\xi,\eta)$ to be employed as the coefficients

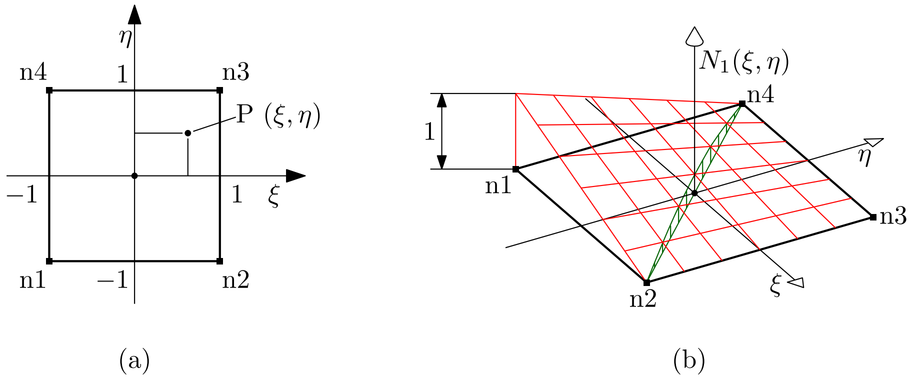

Figure 1.1: Quadrilateral elementary domain (a), and a representative weight function (b).

(weights) of a moving weighted average

$$f(\xi,\eta) \stackrel{\text{def}}{=} \sum_i N_i(\xi,\eta) f_i \tag{1.1}$$

Requisite for such weight functions are that

- the influence of a node is unitary at its location, whereas other nodes influence locally vanishes

$$N_i(\xi_j,\eta_j) = \delta_{ij} \tag{1.2}$$

- for each point of the domain, the sum of the weights is unitary

$$\sum_i N_i(\xi,\eta) = 1,\ \forall[\xi,\eta] \tag{1.3}$$

Moreover, functions should be continuous and straightforwardly differentiable up to any required degree.

Low order polynomials are ideal candidates for the application; for the particular element, the nodal weight functions may be stated as

$$N_i(\xi,\eta) \stackrel{\text{def}}{=} \frac{1}{4}\,(1 \pm \xi)\,(1 \pm \eta)\,, \tag{1.4}$$

where sign ambiguity is resolved for each i-th node by enforcing Eqn. 1.2.

The (1.3) combination of 1.4 turns into a generic linear relation in (ξ, η) with coplanar – but otherwise arbitrary – (ξ_i, η_i, f_i) points.

Further generality may be introduced by *not* enforcing coplanarity.

The weight functions for the four-node quadrilateral are in fact quadratically incomplete[1] in nature due to the presence of the $\xi\eta$ product, and the absence of any ξ^2, η^2 term. Each term, and the combined interpolation function $f(\xi, \eta)$, defined as in Eqn. 1.1, behave linearly if restricted to $\xi =$ const. or $\eta =$ const. loci – namely along the four edges, whereas quadratic behaviour may arise along a general direction, e.g. along the diagonals, as in the Fig. 1.2.1b example. Such behaviour is called *bilinear*.

We now consider the $f(\xi, \eta)$ weight function partial derivatives. The partial derivative

$$\frac{\partial f}{\partial \xi} = \underbrace{\left(\frac{f_2 - f_1}{2}\right)}_{[\Delta f/\Delta\xi]_{12}} \underbrace{\left(\frac{1-\eta}{2}\right)}_{N_1+N_2} + \underbrace{\left(\frac{f_3 - f_4}{2}\right)}_{[\Delta f/\Delta\xi]_{43}} \underbrace{\left(\frac{1+\eta}{2}\right)}_{N_4+N_3} = a\eta + b \tag{1.5}$$

linearly varies from the incremental ratio value measured at the $\eta = -1$ lower edge, to the value measured at the $\eta = 1$ upper edge; the other partial derivative

$$\frac{\partial f}{\partial \eta} = \left(\frac{f_4 - f_1}{2}\right)\left(\frac{1-\xi}{2}\right) + \left(\frac{f_3 - f_2}{2}\right)\left(\frac{1+\xi}{2}\right) = c\xi + d. \tag{1.6}$$

similarly behaves, with $c = a$. However, partial derivatives in ξ, η remain constant along the corresponding differentiation direction [2].

1.2.2 Gaussian quadrature rules for reference domains.

Reference one dimensional domain. The gaussian quadrature rule for approximating the definite integral of a $f(\xi)$ function over the $[-1, 1]$ reference interval is constructed as the customary weighted sum of internal function samples, namely

$$\int_{-1}^{1} f(\xi)d\xi \approx \sum_{i=1}^{n} f(\xi_i)w_i; \tag{1.7}$$

[1] or *enriched linear*, as discussed above and in the following

[2] The relevance of such partial derivative orders will appear clearer to the reader once the strain field will have been derived in paragraph XXX.

Its peculiarity is to employ location-weight pairs (ξ_i, w_i) that are optimal with respect to the polynomial class of functions. Nevertheless, such choice has revealed itself to be robust enough for a more general use.

Let's consider an m-th order polynomial

$$p(\xi) \stackrel{\text{def}}{=} a_m \xi^m + a_{m-1}\xi^{m-1} + ... + a_1 \xi + a_0$$

whose exact integral is

$$\int_{-1}^{1} p(\xi) d\xi = \sum_{j=0}^{m} \frac{(-1)^j + 1}{j+1} a_j$$

The integration residual between the exact definite integral and the weighted sample sum is defined as

$$r\left(a_j, (\xi_i, w_i)\right) \stackrel{\text{def}}{=} \sum_{i=1}^{n} p(\xi_i) w_i - \int_{-1}^{1} p(\xi) d\xi \tag{1.8}$$

The optimality condition is stated as follows: the quadrature rule involving n sample points (ξ_i, w_i), $i = 1 \dots n$ is optimal for the m-th order polynomial if a) the integration residual is null for general a_j values , and b) such condition does not hold for any lower-order sampling rule.

Once observed that the zero residual requirement is satisfied by any sampling rule if the polynomial a_j coefficients are all null, condition a) may be enforced by imposing that such zero residual value remains constant with varying a_j terms, i.e.

$$\left\{ \frac{\partial r\left(a_j, (\xi_i, w_i)\right)}{\partial a_j} = 0, \quad j = 0 \dots m \right. \tag{1.9}$$

A system of $m+1$ polynomial equations of degree $m-1$ is hence obtained in the $2n$ (ξ_i, w_i) unknowns; in the assumed absence of redundant equations, solutions are not allowed if the constraints outnumber the unknowns, i.e. $m > 2n-1$. Limiting our discussion to the threshold condition $m = 2n-1$, an attentive algebraic manipulation of Eqns. 1.9 may be performed in order to extract the (ξ_i, w_i) solutions, which are unique apart from the pair permutations[3].

[3] In this note, location-weight pairs are obtained for the gaussian quadrature

n	ξ_i	w_i
1	0	2
2	$\pm\frac{1}{\sqrt{3}}$	1
3	0	$\frac{8}{9}$
	$\pm\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
4	$\pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18+\sqrt{30}}{36}$
	$\pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18-\sqrt{30}}{36}$

Table 1.1: Integration points for the lower order gaussian quadrature rules.

Eqns. 1.9 solutions are reported in Table 1.1 for quadrature rules with up to $n = 4$ sample points[4].

It is noted that the integration points are symmetrically distributed with respect to the origin, and that the function is never sampled at

rule of order $n = 2$, aiming at illustrating the general procedure. The general $m = 2n - 1 = 3$rd order polynomial is stated in the form

$$p(\xi) = a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0, \quad \int_{-1}^{1} p(\xi)d\xi = \frac{2}{3}a_2 + 2a_0,$$

whereas the integral residual is

$$r = a_3\left(w_1\xi_1^3 + w_2\xi_2^3\right) + a_2\left(w_1\xi_1^2 + w_2\xi_2^2 - \frac{2}{3}\right) + a_1\left(w_1\xi_1 + w_2\xi_2\right) + a_0\left(w_1 + w_2 - 2\right)$$

Eqns 1.9 may be derived as

$$\begin{cases} 0 = \frac{\partial r}{\partial a_3} = w_1\xi_1^3 + w_2\xi_2^3 & (e_1) \\ 0 = \frac{\partial r}{\partial a_2} = w_1\xi_1^2 + w_2\xi_2^2 - \frac{2}{3} & (e_2) \\ 0 = \frac{\partial r}{\partial a_1} = w_1\xi_1 + w_2\xi_2 & (e_3) \\ 0 = \frac{\partial r}{\partial a_0} = w_1 + w_2 - 2 & (e_4) \end{cases}$$

which are independent of the a_j coefficients.

By composing $\left(e_1 - \xi_1^2 e_3\right)/\left(w_2\xi_2\right)$ it is obtained that $\xi_2^2 = \xi_1^2$; e_2 may then be written as $(w_1 + w_2)\xi_1^2 = 2/3$, and then as $2\xi_1^2 = 2/3$, according to e_4. It derives that $\xi_{1,2} = \pm 1/\sqrt{3}$. Due to the opposite nature of the roots, e_3 implies $w_2 = w_1$, relation, this, that turns e_4 into $2w_1 = 2w_2 = 2$, and hence $w_{1,2} = 1$.

[4]see `https://pomax.github.io/bezierinfo/legendre-gauss.html` for higher order gaussian quadrature rule sample points.

the $\{-1, 1\}$ extremal points.

General one dimensional domain. The extension of the one dimensional quadrature rule from the reference domain $[-1, 1]$ to a general $[a, b]$ domain is pretty straightforward, requiring just a change of integration variable to obtain the following

$$\begin{aligned}\int_a^b f(x)dx &= \frac{b-a}{2}\int_{-1}^1 f\left(\frac{b+a}{2} + \frac{b-a}{2}\xi\right) d\xi, \\ &\approx \frac{b-a}{2}\sum_{i=1}^n f\left(\frac{b+a}{2} + \frac{b-a}{2}\xi_i\right) w_i.\end{aligned}$$

Reference quadrangular domain. A quadrature rule for the reference quadrangular domain of Figure XXX may be derived by nesting the quadrature rule defined for the reference interval, see Eqn. 1.7, thus obtaining

$$\int_{-1}^1 \int_{-1}^1 f(\xi, \eta)\, d\xi d\eta \approx \sum_{i=1}^p \sum_{j=1}^q f\left(\xi_i, \eta_j\right) w_i w_j \tag{1.10}$$

where (ξ_i, w_i) and (ξ_j, w_j) are the coordinate-weight pairs of the two quadrature rules of p and q order employed for spanning the two coordinate axes. The equivalent notation

$$\int_{-1}^1 \int_{-1}^1 f(\xi, \eta)\, d\xi d\eta \approx \sum_{l=1}^{pq} f\left(\underline{\xi}_{\,l}\right) w_l \tag{1.11}$$

emphasises the characteristic nature of the pq point/weight pairs for the domain and for the quadrature rule employed; a general integer biiection $\{1 \ldots pq\} \leftrightarrow \{1 \ldots p\} \times \{1 \ldots q\}$, $l \leftrightarrow (i, j)$ e.g.

$$\{i-1; j-1\} = (l-1) \bmod (p, q), \quad l-1 = (j-1)q + (i-1) \tag{1.12}$$

may be utilized to formally derive the two-dimensional quadrature rule pairs

$$\underline{\xi}_{\,l} = \left(\xi_i, \eta_j\right), \quad w_l = w_i w_j, \quad l = 1 \ldots pq \tag{1.13}$$

from their uniaxial counterparts.

General quadrangular domain. The interpolation functions introduced in Paragraph **??** may be be profitably employed in defining a coordinate mapping between a general quadrangular domain – see Fig. XXXx – and its reference counterpart.

In particular, we first define the $\underline{\xi}_i \mapsto \underline{x}_i$ mapping for the coordinates of the four vertices[5] in the natural (or reference) ξ, η and in the physical x, y reference systems. Then, a mapping for the inner points may be derived by interpolation, namely

$$\underline{x}\left(\underline{\xi}\right) = \sum_{i=1}^{4} N_i\left(\underline{\xi}\right)\underline{x}_i \tag{1.14}$$

The availability of an inverse $\underline{\xi}\left(\underline{x}\right)$ mapping is not granted; in particular, a closed form representation for such inverse is not generally available[6].

The rectangular infinitesimal area $dA_{\xi\eta} = d\xi d\eta$ in the neighborhood of a $\xi^\star, \eta^\star$ location, see Figure XXXa, is mapped to the quadrangle of Figure XXXb, which is composed by the two triangular areas

$$\begin{aligned} dA_{xy} = & \frac{1}{2!}\begin{vmatrix} 1 & x\left(\xi^\star, \eta^\star\right) & y\left(\xi^\star, \eta^\star\right) \\ 1 & x\left(\xi^\star + d\xi, \eta^\star\right) & y\left(\xi^\star + d\xi, \eta^\star\right) \\ 1 & x\left(\xi^\star, \eta^\star + d\eta\right) & y\left(\xi^\star, \eta^\star + d\eta\right) \end{vmatrix} + \\ & + \frac{1}{2!}\begin{vmatrix} 1 & x\left(\xi^\star + d\xi, \eta^\star + d\eta\right) & y\left(\xi^\star + d\xi, \eta^\star + d\eta\right) \\ 1 & x\left(\xi^\star, \eta^\star + d\eta\right) & y\left(\xi^\star, \eta^\star + d\eta\right) \\ 1 & x\left(\xi^\star + d\xi, \eta^\star\right) & y\left(\xi^\star + d\xi, \eta^\star\right) \end{vmatrix} \end{aligned} \tag{1.15}$$

where the determinant formula for the area of a triangle, shown below along with its n-dimensional symplex hypervolume generalization

$$\mathcal{A} = \frac{1}{2!}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}, \quad \mathcal{H} = \frac{1}{n!}\begin{vmatrix} 1 & \underline{x}_1 \\ 1 & \underline{x}_2 \\ \vdots & \vdots \\ 1 & \underline{x}_{n+1} \end{vmatrix} \tag{1.16}$$

is employed. By operating a local multivariate linearization of 1.15 matrix terms,

[5]The condensed notation $\underline{\xi}_i \equiv (\xi_i, \eta_i)$, $\underline{x}_i \equiv (x_i, y_i)$ for coordinate vectors is employed.

[6]For a given $\bar{x}$ physical point, however, Newton-Raphson iterations rapidly converge to the $\underline{\xi}\left(\underline{\bar{x}}\right)$ solution if the centroid is supplied for algorithm initialization, see Section XXX

$$
\begin{aligned}
dA_{xy} \approx \quad & \frac{1}{2!}\begin{vmatrix} 1 & x^\star & y^\star \\ 1 & x^\star + x^\star_{,\xi} d\xi & y^\star + y^\star_{,\xi} d\xi \\ 1 & x^\star + x^\star_{,\eta} d\eta & y^\star + y^\star_{,\eta} d\eta \end{vmatrix} + \\
& + \frac{1}{2!}\begin{vmatrix} 1 & x^\star + x^\star_{,\xi} d\xi + x^\star_{,\eta} d\eta & y^\star + y^\star_{,\xi} d\xi + y^\star_{,\eta} d\eta \\ 1 & x^\star + x^\star_{,\eta} d\eta & y^\star + y^\star_{,\eta} d\eta \\ 1 & x^\star + x^\star_{,\xi} d\xi & y^\star + y^\star_{,\xi} d\xi \end{vmatrix}
\end{aligned}
$$

is obtained, where $x^\star, y^\star$ and $x^\star_{,\xi}, x^\star_{,\eta}, y^\star_{,\xi}, y^\star_{,\eta}$ are the x, y functions and their first order partial derivatives, as sampled at the $\xi^\star, \eta^\star$ point; infinitesimal terms of higher order than $d\xi d\eta$ are neglected .

After some matrix manipulations[7] the following expression is obtained

$$
dA_{xy} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & x^\star_{,\xi} & y^\star_{,\xi} \\ 0 & x^\star_{,\eta} & y^\star_{,\eta} \end{vmatrix} d\xi d\eta = \underbrace{\begin{vmatrix} x^\star_{,\xi} & y^\star_{,\xi} \\ x^\star_{,\eta} & y^\star_{,\eta} \end{vmatrix}}_{|J^{\mathrm{T}}(\xi,\eta)|} dA_{\xi\eta} \tag{1.17}
$$

that equates the ratio of the mapped and the original areas to the determinant of the transformation (transpose) Jacobian matrix[8].

Once carried out the preparatory passages, we obtain

$$
\iint_{A_{xy}} g(x,y) dA_{xy} = \iint_{-1}^{1} g\left(x\left(\xi,\eta\right), y\left(\xi,\eta\right)\right) \left|J(\xi,\eta)\right| d\xi d\eta, \tag{1.18}
$$

thus reducing the quadrature over a general domain to its reference domain counterpart, which has been discussed in the paragraph above.

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[7] For both the determinants, the first column is multiplied by $x^\star$ and subtracted to the second column, and then subtracted to the third column once multiplied by $y^\star$. The first row is then subtracted to the others. On the second determinant alone, both the second and the third columns are changed in sign; then, the second and the third rows are summed to the first. The two determinants are now formally equal, and the two $1/2$ multipliers are summed to unity. The $d\xi$ and the $d\eta$ factors may then be collected from the second and the third rows, respectively.

[8] The Jacobian matrix for a general $\underline{\xi} \mapsto \underline{\mathrm{x}}$ mapping is in fact defined according to

$$
[J(\underline{\xi}^\star)]_{ij} \stackrel{\mathrm{def}}{=} \left.\frac{\partial x_i}{\partial \xi_j}\right|_{\underline{\xi}=\underline{\xi}^\star} \qquad i,j = 1 \ldots n
$$

being i the generic matrix term row index, and j the column index

Based on Eqn. 1.11, the quadrature rule

$$\iint_{A_{xy}} g(\underline{\mathrm{x}})dA_{xy} \approx \sum_{l=1}^{pq} g\left(\underline{\mathrm{x}}\left(\underline{\xi}_{l}\right)\right)\left|J(\underline{\xi}_{l})\right| w_l \tag{1.19}$$

is derived, stating that the definite integral of a g integrand over a quadrangular domain pertaining to a physical x, y plane (x, y may be dimensional quantities, namely lengths) may be approximated by

1. defining a reference to physical domain mapping based on vertex physical coordinates interpolation;

2. sampling the function at physical locations which are the images of the Gaussian integration points obtained for the reference domain;

3. proceeding with a weighted sum of the collected samples, where the weights consist in the product of the adimensional w_l Gauss point weight (suitable for integrating on the reference domain), and of a dimensional area scaling term consisting in the determinant of the transformation Jacobian matrix, as evaluated locally at the Gauss point.

1.3 Basic theory of plates

P displacement components as a function of the Q reference point motion.

$$u_P = u + z\,(1+\tilde{\epsilon}_z)\,sin\,\phi \tag{1.20}$$
$$v_P = v - z\,(1+\tilde{\epsilon}_z)\,sin\,\theta \tag{1.21}$$
$$w_P = w + z\,((1+\tilde{\epsilon}_z)\,cos(\phi)\,cos(\theta) - 1) \tag{1.22}$$

$$\tilde{\epsilon}(z) = \frac{1}{z}\int_0^z \epsilon_z d\varsigma \tag{1.23}$$
$$= \frac{1}{z}\int_0^z \left(-\nu\epsilon_x - \nu\epsilon_y\right) d\varsigma \tag{1.24}$$

P displacement components as a function of the Q reference point motion, linarized with respect to the small rotations and small strain hypotheses.

$$u_P = u + z\phi \tag{1.25}$$
$$v_P = v - z\theta \tag{1.26}$$
$$w_P = w \tag{1.27}$$

Relation between the normal displacement x, y gradient (i.e. the deformed plate slope), the rotations and the out-of-plane, interlaminar, averaged shear strain components.

$$\frac{\partial w}{\partial x} = \bar{\gamma}_{zx} - \phi \tag{1.28}$$
$$\frac{\partial w}{\partial y} = \bar{\gamma}_{yz} + \theta \tag{1.29}$$

Strains at point P.

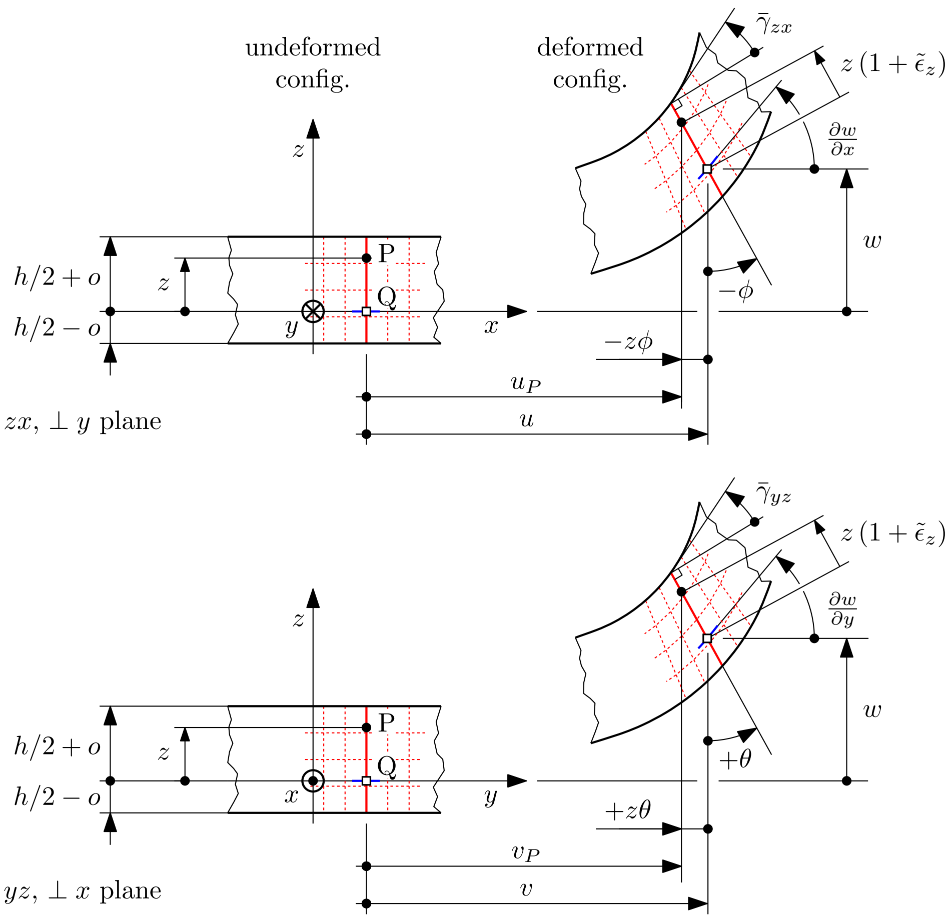

Figure 1.2: Relevant dimensions for describing the deformable plate kinematics.

$$\epsilon_x = \frac{\partial u_P}{\partial x} = \frac{\partial u}{\partial x} + z\frac{\partial \phi}{\partial x} \tag{1.30}$$

$$\epsilon_y = \frac{\partial v_P}{\partial y} = \frac{\partial v}{\partial y} - z\frac{\partial \theta}{\partial y} \tag{1.31}$$

$$\gamma_{xy} = \frac{\partial u_P}{\partial y} + \frac{\partial v_P}{\partial x} \tag{1.32}$$

$$= \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + z\left(+\frac{\partial \phi}{\partial y} - \frac{\partial \theta}{\partial x}\right) \tag{1.33}$$

Generalized plate strains: membrane strains.

$$\underline{\bar{\epsilon}} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} \bar{\epsilon}_x \\ \bar{\epsilon}_y \\ \bar{\gamma}_{xy} \end{pmatrix} \tag{1.34}$$

Generalized plate strains: curvatures.

$$\underline{\kappa} = \begin{pmatrix} +\frac{\partial \phi}{\partial x} \\ -\frac{\partial \theta}{\partial y} \\ +\frac{\partial \phi}{\partial y} - \frac{\partial \theta}{\partial x} \end{pmatrix} = \begin{pmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix} \tag{1.35}$$

Compact form for the strain components at P.

$$\underline{\epsilon} = \underline{\bar{\epsilon}} + z\,\underline{\kappa} \tag{1.36}$$

Hook law for an isotropic material, under plane stress conditions.

$$\underline{\underline{\mathrm{D}}} = \frac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \tag{1.37}$$

Normal components for stress and strain, the latter for the isotropic material case only.

$$\sigma_z = 0 \tag{1.38}$$

$$\epsilon_z = -\nu\left(\epsilon_x + \epsilon_y\right) \tag{1.39}$$

Stresses at P.

$$\underline{\sigma} = \underline{\underline{\mathrm{D}}}\,\underline{\epsilon} = \underline{\underline{\mathrm{D}}}\,\underline{\bar{\epsilon}} + z\,\underline{\underline{\mathrm{D}}}\,\underline{\kappa} \tag{1.40}$$

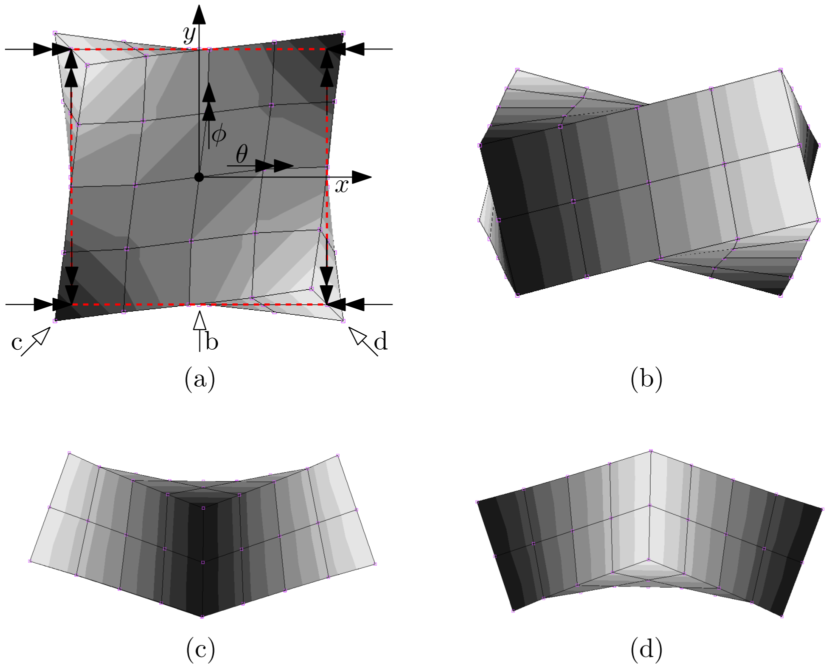

Figure 1.3: Positive κ_{xy} torsional curvature for the plate element. Subfigure (a) shows the positive γ_{xy} shear strain at the upper surface, the (in-plane) undeformed midsurface, and the negative γ_{xy} at the lower surface; the point of sight related to subfigures (b) to (d) are also evidenced. θ and ϕ rotation components decrease with x and increase with y, respectively, thus leading to positive κ_{xy} contributions. As shown in subfigures (c) and (d), the torsional curvature of subfigure (b) evolves into two anticlastic bending curvatures if the reference system is aligned with the square plate element diagonals, and hence rotated by 45° with respect to z.

Membrane (direct and shear) stress resultants (shear flows).

$$\underline{q} = \begin{pmatrix} q_x \\ q_y \\ q_{xy} \end{pmatrix} = \int_h \underline{\sigma}\, dz \tag{1.41}$$

$$= \underbrace{\int_h \underline{\underline{D}}\, dz}_{\underline{\underline{A}}}\ \bar{\underline{\epsilon}} + \underbrace{\int_h \underline{\underline{D}}\, z dz}_{\underline{\underline{B}}}\ \underline{\kappa} \tag{1.42}$$

Bending and torsional moment stress resultants (moment flows).

$$\underline{m} = \begin{pmatrix} m_x \\ m_y \\ m_{xy} \end{pmatrix} = \int_h \underline{\sigma}\, dz \tag{1.43}$$

$$= \underbrace{\int_h \underline{\underline{D}}\, z dz}_{\underline{\underline{B}} \equiv \underline{\underline{B}}^{\mathrm{T}}}\ \bar{\underline{\epsilon}} + \underbrace{\int_h \underline{\underline{D}}\, z^2 dz}_{\underline{\underline{C}}}\ \underline{\kappa} \tag{1.44}$$

Cumulative generalized strain - stress relations for the plate (or for the laminate)

$$\begin{pmatrix} \underline{q} \\ \underline{m} \end{pmatrix} = \begin{pmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{B}}^{\mathrm{T}} & \underline{\underline{C}} \end{pmatrix} \begin{pmatrix} \bar{\underline{\epsilon}} \\ \underline{\kappa} \end{pmatrix} \tag{1.45}$$

Hook law for the orthotropic material in plane stress conditions, with respect to principal axes of orthotropy;

$$\underline{\underline{D}}_{123} = \begin{pmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{pmatrix} \tag{1.46}$$

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{pmatrix} = \underline{\underline{T}}_1 \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} \qquad \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{pmatrix} = \underline{\underline{T}}_2 \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} \tag{1.47}$$

where

$$\underline{\underline{T}}_1 = \begin{pmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{pmatrix} \tag{1.48}$$

$$\underline{\underline{T}}_2 = \begin{pmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{pmatrix} \tag{1.49}$$

α is the angle between 1 and x;

$$m = cos(\alpha) \qquad\qquad n = sin(\alpha) \tag{1.50}$$

The inverse transformations may be obtained based on the relations

$$\underline{\underline{T}}_1^{-1}(+\alpha) = \underline{\underline{T}}_1(-\alpha) \qquad\qquad \underline{\underline{T}}_2^{-1}(+\alpha) = \underline{\underline{T}}_2(-\alpha) \tag{1.51}$$

Finally

$$\underline{\sigma} = \underline{\underline{D}}\,\underline{\epsilon} \qquad\qquad \underline{\underline{D}} \equiv \underline{\underline{D}}_{xyz} = \underline{\underline{T}}_1^{-1}\underline{\underline{D}}_{123}\underline{\underline{T}}_2 \tag{1.52}$$

Notes:

- Midplane is ill-defined if the material distribution is not symmetric; the geometric midplane (i.e. the one obtained by ignoring the material distribution) exhibits no relevant properties in general. Its definition is nevertheless pretty straighforward.
- If the unsimmetric laminate is composed by isotropic layers, a reference plane may be obtained for which the $\underline{\underline{B}}$ membrane-to-bending coupling matrix vanishes; a similar condition may not be verified in the presence of orthotropic layers.
- Thermally induced distortion is not self-compensated in an unsymmetric laminate even if the temperature is held constant through the thickness.

Bibliography

[1] A. E. H. Love, *A treatise on the mathematical theory of elasticity.* Cambridge university press, 2013.